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RANDOM TREES UNDER CH

BY

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ABSTRACT

We extend Jensen's Theorem that Souslin's Hypothesis is consistent with CH, by showing that the statement *Souslin's Hypothesis holds in any forcing extension by a measure algebra* is consistent with CH. We also formulate a variation of the principle $(*)$ (see [AT97], [Tod00]) for closed sets of ordinals, and show its consistency relative to the appropriate large cardinal hypothesis. Its consistency with CH would extend Silver's Theorem that, assuming the existence of an inaccessible cardinal, the failure of Kurepa's Hypothesis is consistent with CH, by its implication that the statement *Kurepa's Hypothesis fails in any forcing extension by a measure algebra* is consistent with CH.

1. Introduction

Souslin's Hypothesis that the real line is, up to an isomorphism, the only totally ordered ccc continuum has played an important role in the development of set theory. Its independence from the usual Zermelo–Frankel axioms of set theory, as established by Solovay and Tennenbaum (see [ST71]), led to the development of an even more important additional axiom: Martin's Axiom. The independence of SH from CH was considerably more difficult and was established by Jensen several years after the Solovay and Tennenbaum result (see [DJ74]).

In [Lav87] Laver extends the results of Solovay and Tennenbaum by showing that not only does $MA_{\aleph_1}$ imply SH, but it implies RSH where RSH is the

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statement *there are no Souslin trees in any forcing extension by a measure algebra*, thus establishing the independence of SH from the classical hypothesis that the Lebesgue measure can be extended to all sets of reals. In this paper we establish the corresponding version of Jensen's result; in other words we prove that for a suitable model of CH, Souslin's Hypothesis remains true in any forcing extension by a measure algebra, thus establishing the independence of SH from statements like $\mathfrak{b} = \aleph_1 < \mathfrak{c}$. More precisely, we show that that the principle $(*)$ of Abraham–Todorčević ([AT97], [Tod00]) implies that SH holds in any forcing extension by a measure algebra. This gives a negative answer to a question of Laver (private communication): whether extending a model of CH by a random real adds a Souslin tree.

The principle $(*)$, in its most general (and optimal) form (see [Tod00]), is:

$(*)$ For every P-ideal $\mathcal{I}$ of countable subsets of some set S, either

(1) there is an uncountable $A \subseteq S$ such that $A^{[\aleph]} \subseteq \mathcal{I}$,

or

(2) S can be decomposed into countably many sets orthogonal to $\mathcal{I}$

(where $A^{[\aleph]}$ denotes the set of all countably infinite subsets of A). Note that in the original version of $(*)$, $S = \omega_1$, and this is all we use in the consideration of Souslin trees in the forcing extension by some measure algebra. Let us interject here to recall the relevant definitions. An **ideal** on a set S is a downwards closed family of subsets of S which is closed under pairwise unions. We always assume that an ideal on S consists only of countable subsets of S, and moreover that it includes all finite subsets of S. A ***P*-ideal** is an ideal $\mathcal{I}$ where every countable subset of $\mathcal{I}$ has an upper $\subseteq^*$-bound in $\mathcal{I}$. A subset X of S is **orthogonal** to a family $\mathcal{F}$ of subsets of S, if $X \cap F$ is finite for all $F \in \mathcal{F}$.

We shall in fact prove corresponding results for the stronger statement SH^+: *all Aronszajn trees are special* which was originally deduced from $\mathrm{MA}_{\aleph_1}$ by Baumgartner, Malitz and Reinhardt [BMR70]. Laver [Lav87], in fact, proves that $\mathrm{MA}_{\aleph_1}$ implies RSH^+ where RSH^+ is the statement *all Aronszajn trees are special in any forcing extension by a measure algebra.*

Since Jensen [DJ74], in fact, proves that SH^+ is consistent with CH, it is natural to try and strengthen this to the statement that RSH^+ is consistent with CH. Indeed this is what we attempt to do, and moreover we apply these ideas to randomize the notion of a square sequence. However, for this we need a variation of $(*)$ which is introduced in Section 2 as $(\star_c)$; and the proof of the consistency of this variation with CH is ongoing work of the author (see Section 2.1.1 for further discussion).

The independence of Kurepa's Hypothesis that there exists a Kurepa tree was first proved by Silver in [Sil71], where it is moreover shown that assuming the existence of an inaccessible cardinal $\neg$KH is consistent with GCH. (He also showed that the inaccessible is necessary for $\neg$KH.) In [Dev78] and [Dev80] Devlin extends Silver's result by constructing a model of $\neg$KH + SH + GCH from an inaccessible cardinal. In [Dev83], is a result of Baumgartner that PFA: the Proper Forcing Axiom, a strengthening of $\mathrm{MA}_{\aleph_1}$, implies that every tree of size and height ω_1 is essentially special, and thus in particular, PFA implies $\neg\mathrm{KH}^-$ where KH^- is the weakening of KH which states that there exists an ω_1-tree which is not essentially special. In an unpublished piece of work [Bau], Baumgartner showed further that PFA implies that every tree of size and height ω_1 is essentially special in any extension by a measure algebra, and in particular, PFA implies $\neg\mathrm{RKH}^-$ where RKH^- is the random version of the statement KH^-. In this note we extend Devlin's result (and hence Silver's result) along the lines of Baumgartner, by proving that assuming the existence of an inaccessible cardinal the conjunction of the statements RSH^+ and $\neg\mathrm{RKH}^-$ is consistent with CH, thereby giving a positive answer to a question of Baumgartner. However, this once again is dependent upon proving the consistency of $(\star_c)$ with CH.

Finally, we comment on the choice to force with a measure algebra rather than any of the other algebras. Recall that it is a Theorem of Shelah [She84] that SH fails in any forcing extension by one Cohen real. A simpler and elegant construction of a Souslin tree from a Cohen real is given by Todorčević in [Tod87]. Moreover, as shown by Carlson and Laver [CL89], if V satisfies PFA and s is a Sacks real over V, then $V[s]$ satisfies $\mathrm{MA}_{\aleph_1}$ (and in particular SH); however, adding a Sacks real to a model of CH introduces a diamond sequence, and in particular a Souslin tree; hence, Sacks reals and random reals differ in this regard. This clearly gives further motivation for Laver's question above.

Furthermore, by random forcing we obtain new independence results of SH from various cardinal invariants (see [vD84] for an introduction to cardinal invariants). For example, all of the previously known models of SH satisfied either CH or $\mathfrak{b} > \aleph_1$. By applying our results to a measure algebra of weight at least $\aleph_2$ we obtain the following Corollary*:

Corollary 1.1: SH *is independent of* $\mathfrak{d} = \aleph_1 < \mathfrak{c}$.

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* *Added in proof.* Since this paper was first submitted, the same result was obtained independently by Mildenberger–Shelah [MS03].

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2. The principle $(\star_c)$

Our interest in $(*)$ lies in the fact that it implies certain consequences of PFA—for example, see one of the main results of this paper: Theorem 3.17—and yet is consistent with CH (see [AT97], [Tod00], resp.):

THEOREM 2.1 (Abraham–Todorčević): $(*)$ *for* $S = \omega_1$ *is relatively consistent with* CH.

THEOREM 2.2 (Todorčević): *Assuming that the existence of a supercompact cardinal is consistent,* $(*)$ *is consistent with* CH.

The principle $(\star_c)$ is a variation of $(*)$, where "uncountable" is strengthened to "closed uncountable", and "countable decomposition" is weakened to "stationary":

Definition 2.3:

$(\star_c)$ For any ordinal θ of uncountable cofinality, for every P-ideal $\mathcal{I}$ of countable subsets of θ, either

(1) there is a closed uncountable $C \subseteq \theta$ such that $C^{[\aleph]} \subseteq \mathcal{I}$,

or

(2) there is a stationary subset of θ orthogonal to $\mathcal{I}$.

The principle $(\star_c)$ is false without the restriction on the cofinality of θ, but of course this not a significant restriction. Furthermore, for our application to random Aronszajn trees we only consider the case $\theta = \omega_1$. Our interest in the variation $(\star_c)$ is that it entails consequences of PFA which do not seem to follow from $(*)$. For example, $(*)$ does not appear to imply SH^+, while we shall see that $(\star_c)$ implies SH^+ and more.

As with the dichotomy $(*)$, the dichotomy $(\star_c)$ is a consequence of PFA (Theorem 4.7). The special case of $(\star_c)$ where $\theta = \omega_1$ is equiconsistent with ZFC. However, in [Tod87, (1.10)] (see also [Jen72]), it is shown that if there is no square sequence on $\aleph_2$ (see Section 3.1) then $\aleph_2$ is weakly compact in L. Hence by Theorem 3.21, $(\star_c)$ for $\theta \leq \omega_2$ requires a weakly compact cardinal. Conversely, the restriction of $(\star_c)$ to ω_2 is equiconsistent with a weakly compact cardinal. Moreover, $\neg\square(\aleph_2) + \neg\square_{\aleph_2}$ (see Section 3.1) implies that $0^\sharp$ exists (see [Tod02]). Hence by Theorem 3.21 the restriction of $(\star_c)$ to $\theta \leq \omega_3$ implies that

$0^\sharp$ exists, and therefore has considerably higher consistency strength than $(\star_c)$ for $\theta \leq \omega_2$. By recent results in Inner Model Theory (see [Kan94]) we obtain as a corollary of Theorem 3.21 that the unrestricted principle $(\star_c)$ has even far more large cardinal strength.

For the remainder of this section we examine the optimality of $(\star_c)$, i.e. we consider strengthenings and variations of $(\star_c)$ and prove that they are inconsistent; and, most importantly, we discuss the consistency of $(\star_c)$ with CH.

2.1. Club sequences. The principle $(\star_c)$ is optimal in the sense that in alternative $(\star_c)(1)$ we cannot strengthen "closed uncountable" to "closed unbounded". A **club-guessing sequence** on an ordinal θ is a sequence

$$\langle C_\alpha : \alpha < \theta, \mathrm{cf}(\alpha) = \aleph \rangle$$

such that

(a) C_α is an unbounded subset of α of order type ω,

(b) for every closed unbounded $C \subseteq \theta$, there is an α such that $C_\alpha \subseteq C$.

We provide a proof of the following result of Shelah (see [She94]) for the reader's convenience.

Theorem 2.4 (Shelah): *There is a club-guessing sequence on ω_2.*

Proof: Suppose by way of contradiction that there is no such object. We shall construct closed unbounded sets $E^\xi \subseteq \omega_2$ and sequences

$$\langle f^\xi_\alpha : \alpha \in E^\xi, \mathrm{cf}(\alpha) = \aleph \rangle$$

where $f^\xi_\alpha \colon \omega \to \alpha$ is a nondecreasing cofinal map, by recursion on $\xi < \omega_1$, such that for all α,

$$f^\xi_\alpha(n) \leq f^\gamma_\alpha(n) \quad \text{for all } n < \omega, \text{ for all } \gamma < \xi, \tag{1}$$

as described below.

To start the recursion put $E^0 = \omega_2$ and let $\langle f^\xi_\alpha \rangle$ be arbitrary. Given $\xi < \omega_1$ assume that E^γ and $\langle f^\gamma_\alpha \rangle$ have been defined for all $\gamma < \xi$. First suppose that ξ is a successor. By assumption there is a club $D^\xi \subseteq \omega_2$ such that

$$\mathrm{ran}(f^{\xi-1}_\alpha) \setminus \{0\} \not\subseteq D^\xi \quad \text{for all } \alpha \in E^{\xi-1} \text{ with } \mathrm{cf}(\alpha) = \aleph. \tag{2}$$

Choose a club $E^\xi \subseteq E^{\xi-1}$ such that $D^\xi \cap \alpha$ is unbounded in α for all $\alpha \in E^\xi$. Then for each $\alpha \in E^\xi$ with $\mathrm{cf}(\alpha) = \aleph$, by defining $f^\xi_\alpha \colon \omega \to \alpha$ by

$$f^\xi_\alpha(n) = \begin{cases} \max(D^\xi \cap (f^{\xi-1}_\alpha(n)+1)) & \text{if } \min(D^\xi) \leq f^{\xi-1}_\alpha(n), \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

we obtain a nondecreasing unbounded function, such that

$$f_\alpha^\xi(n) < f_\alpha^{\xi-1}(n) \quad \text{for some } n. \tag{4}$$

If ξ is a limit, find a club $E^\xi \subseteq \bigcap_{\gamma<\xi} E^\gamma$ such that $\bigcap_{\gamma<\xi} D^\gamma \cap \alpha$ is unbounded in α for all $\alpha \in E^\xi$. Then by (3) and by the way we define f_α^γ for γ a limit,

$$f_\alpha^\xi(n) = \min_{\gamma<\xi} f_\alpha^\gamma(n) \quad \text{for all } n$$

defines a nondecreasing cofinal function for all $\alpha \in E^\xi$ with $\mathrm{cf}(\alpha) = \aleph$.

Having completed the recursion, let $E = \bigcap_{\xi<\omega_1} E^\xi$. If we pick any $\alpha \in E$ with $\mathrm{cf}(\alpha) = \aleph$, then from (1) and (4) we arrive at the impossibility that $(f_\alpha^{\xi+1} : \xi < \omega_1)$ is strictly decreasing with respect to $<$. ∎

So let $\vec{C}$ be a club-guessing sequence on ω_2. We associate an ideal with $\vec{C}$ as follows:

Definition 2.5: Let $\mathcal{I}_{\vec{C}}$ be the ideal of all countable $\Omega \subseteq \omega_2$ such that

$$\Omega \perp \vec{C},$$

i.e. $\Omega \cap C_\alpha$ is finite for all $\alpha < \omega_2$ of cofinality $\aleph$.

LEMMA 2.6: *$\mathcal{I}_{\vec{C}}$ is a P-ideal.*

Proof: Suppose that $\{\Omega_n\}_{n=0}^\infty \subseteq \mathcal{I}_{\vec{C}}$. Put $\Omega_\omega = \bigcup_{n=0}^\infty \Omega_n$.

CLAIM 2.7: *For every countable $\Omega \subseteq \omega_2$, $\Omega \cap C_\alpha$ is finite for cocountably many α.*

Proof: Recall that by a **limit point** of a set S of ordinals, we mean a limit point, also called an accumulation point, in the topological sense. Thus α is a limit point of S iff α is a limit ordinal and $S \cap \alpha$ is unbounded in α. Notice that for any α, if $\Omega \cap C_\alpha$ is infinite then α is a limit point of Ω. Since a countable set has countably many limit points, this proves the Claim. ∎

This proves that $\vec{C}$ is **locally countable**, i.e. it has a countable trace on every countable subset of ω_2. Hence there exists a $\Omega \subseteq \Omega_\omega$ which interpolates the pregap formed by the two countable orthogonal families

$$\{\Omega_n\}_{n=0}^\infty \quad \text{and} \quad \{\Omega \cap C_\alpha : \mathrm{cf}(\alpha) = \aleph\}.$$

Then $\Omega \in \mathcal{I}_{\vec{C}}$ and we are done. ∎

Alternative $(\star_c)(2)$ fails dramatically for this P-ideal:

LEMMA 2.8: *There is no set of order type ω^2 orthogonal to $\mathcal{I}_{\vec{C}}$.*

Proof: Let Ω be a subset of ω_2 with $\mathrm{otp}(\Omega) = \omega^2$. By Claim 2.7 we can find an enumeration $\{\alpha_i\}_{i=0}^{\infty}$ of the set of all α such that $\Omega \cap C_\alpha$ is infinite. Choose $\{\xi_j\}_{j=0}^{\infty} \subseteq \Omega$ recursively so that

$$\xi_j \notin \bigcup_{i<j} C_{\alpha_i} \cup \{\xi_i\} \quad \text{for all } j. \tag{5}$$

Note that the order type of Ω is large enough to make this possible. It is clear that $\{\xi_j\} \in \mathcal{I}_{\vec{C}}$ as wanted. ∎

On the other hand, it is immediate from the definition of a club-guessing sequence that there is no closed unbounded subset of θ all of whose infinite subsets are in the ideal. Thus alternative $(\star_c)(1)$ cannot be strengthened to an existential statement about a closed unbounded set, even if the second alternative is severely weakened.

As a counterpoint, we observe that $(\star_c)$ negates even a rather weak consequence of the diamond principle on ω_1:

THEOREM 2.9: $(\star_c)$ *restricted to ω_1 implies that there is no club-guessing sequence on ω_1.*

Proof: Suppose that there is a club-guessing sequence on ω_1. Then if we let $\mathcal{J}$ be the analogous ideal to $\mathcal{I}_{\vec{C}}$, the proofs of Lemma 2.6 and Lemma 2.8 tell us that $\mathcal{J}$ is a P-ideal such that no subset of ω_1 of order type ω^2 is orthogonal to $\mathcal{J}$. Therefore, since there is no closed unbounded subset C of ω_1 such that $C^{[\aleph]} \subseteq \mathcal{J}$, the principle $(\star_c)$ with $\theta = \omega_1$ fails. ∎

2.1.1. Consistency of $(\star_c)$ *with* CH. The consistency of $(\star_c)$ with CH is the subject of some of the author's current research. We make the following conjecture.

CONJECTURE 1: *The conjunction of* CH *and* $(\star_c)$ *for $\theta = \omega_1$, for $\theta = \omega_2$ and for all θ with uncountable cofinality are relatively consistent with* ZFC*, the existence of a weakly compact cardinal and the existence of a supercompact cardinal, respectively.*

Let us discuss the difficulties in proving this. The forcing notion in [Tod00] is shown to be α-proper for all $\alpha < \omega_1$. However, Theorem 2.9 states that the nonexistence of a club-guessing sequence on ω_1 is a consequence of $(\star_c)$ for $\theta = \omega_1$. And as pointed out by Shelah [She98, pp. 854–855] it is easy to see that a club-guessing sequence on ω_1 is preserved by any ω-proper poset. Thus if one begins with a model satisfying GCH with a club-guessing sequence on ω_1, and forces with one of the posets from [Tod00], either for forcing $(*)$ for $\theta = \omega_1$ or for forcing $(*)$ from a supercompact, then one still has a club-guessing sequence on ω_1 in the extension. In particular, this proves that $(*)$ does not entail $(\star_c)$.

Since α-properness is used (in conjunction with another property) to ensure that the iterated forcing does not add reals, we need some other property in place of α-properness to force $(\star_c)$ without adding reals. Shelah has developed a property—roughly speaking, finite powers of any intermediate stage of the iteration must be proper—precisely for the purpose of destroying all club-guessing sequences on ω_1 without violating CH (see [She98, Ch. XVIII, §2]). However, this property seems too strong to allow for $(\star_c)$. Indeed, Shelah states there that it does not even appear possible to specialize all Aronszajn trees with an iteration satisfying this property.

There is only one other known property, called **$\mathfrak{p}$-properness**, which ensures that iterations do not add reals (see [She00]). This is a requirement that all of the posets in the iteration are proper with respect to some fixed parameter $\mathfrak{p}$. While there is a natural choice of parameter for a given P-ideal, we did not see any reason for the existence of a parameter which would work for all ideals appearing in the iteration. Therefore, it seems necessary to develop new theory for iterated forcing in order to prove the consistency of $(\star_c)$ with CH.

2.2. Costationary sets. It is easy to see that $(\star_c)$ is also optimal in the sense that alternative $(\star_c)(2)$ cannot be strengthened by improving "stationary subset of θ" to "countable decomposition of θ", or equivalently, the first alternative of $(*)$ cannot be improved by strengthening "uncountable" to "closed uncountable". Considering the case $\theta = \omega_1$, let $T \subseteq \omega_1$ be an uncountable costationary subset of ω_1. Define an ideal

$$\mathcal{I}_T = T^{[\leq\aleph]} \cup \mathrm{FIN}_{\omega_1}, \tag{i}$$

i.e. $\mathcal{I}_T$ is the ideal generated by the set of all countable subsets of T. Since $T^{[\leq\aleph]}$ is σ-closed, it follows that $\mathcal{I}_T$ is a P-ideal. However, there is no closed uncountable $C \subseteq \omega_1$ with $C^{[\aleph]} \subseteq \mathcal{I}_T$ because T is costationary. And there is no countable decomposition of ω_1 into sets orthogonal to $\mathcal{I}_T$ because T is uncountable.

2.3. The dual of $(\star_c)$. The natural dualization of the principle $(\star_c)$ would be the statement obtained by switching "closed uncountable" with "stationary". However, even if θ is restricted to ω_1 this statement is inconsistent. Indeed, the following variation of $(*)$ for $\theta = \omega_1$, which is formally weaker than the dual of $(\star_c)$, is not consistent with ZFC:

($\bigstar$) For every P-ideal $\mathcal{I}$ of countable subsets of ω_1, either

(1) there is an uncountable $X \subseteq \omega_1$ such that $X^{[\aleph]} \subseteq \mathcal{I}$,

or

(2) there is a closed unbounded subset of ω_1 orthogonal to $\mathcal{I}$.

This is seen by considering the following P-ideal. Let $S_0 \supseteq S_1 \supseteq \cdots$ be a sequence of stationary subsets of ω_1 such that $\bigcap_{n=0}^{\infty} S_n = \emptyset$. Let $\mathcal{I}_{\vec{S}}$ be the ideal of all countable subsets Ω of ω_1 such that

$$\Omega \subseteq^* S_n \quad \text{for all } n. \tag{ii}$$

Claim 2.10: *$\mathcal{I}_{\vec{S}}$ is a P-ideal.*

Proof: Take $\Omega_n \in \mathcal{I}_{\vec{S}}$ ($n \in \mathbb{N}$). Then

$$\bigcup_{n=0}^{\infty} \Omega_n \cap S_n \tag{6}$$

is a member of $\mathcal{I}_{\vec{S}}$ which almost includes every Ω_n. ∎

Claim 2.11: *There is no uncountable $X \subseteq \omega_1$ for which $X^{[\aleph]} \subseteq \mathcal{I}_{\vec{S}}$.*

Proof: Given such an X we could find a countable set A such that $X \setminus A \subseteq \bigcap_{n=0}^{\infty} S_n$. ∎

Claim 2.12: *There is no closed unbounded set orthogonal to $\mathcal{I}_{\vec{S}}$.*

Proof: Let $C \subseteq \omega_1$ be a club. For each n, choose $\alpha_n \in S_n \cap C$. Then $\{\alpha_n : n \in \mathbb{N}\}$ is an infinite subset of C which is in $\mathcal{I}_{\vec{S}}$. ∎

2.4. Club variations of (A^*). Let us observe now that $(\star_c)$ is optimal in the sense that the P-ideal hypothesis is needed. It is even more essential here than for $(*)$: The combinatorial principle (A^*) is a dichotomy with precisely the same alternatives as the principle $(*)$, the difference being that (A^*) applies to $\aleph_1$-generated ideals rather than P-ideals. This is a very strong consequence of PFA that negates CH. Indeed the principle $(*)$ was derived from (A^*) in order to obtain a similar principle compatible with CH (see [AT97]). In our context one may ask whether a variation of (A^*) along the lines of $(\star_c)$ is possible. It is not, as is seen by phrasing a weak version of such a variation:

($\mathrm{A}^{\star_c}$) For every $\aleph_1$-generated ideal $\mathcal{I}$ of countable subsets of ω_1, either

(1) there is an closed unbounded $C \subseteq \omega_1$ such that $C^{[\aleph]} \subseteq \mathcal{I}$,

or

(2) there is an uncountable subset of ω_1 orthogonal to $\mathcal{I}$.

To prove that ($\mathrm{A}^{\star_c}$) is inconsistent, let $\{T_n\}_{n=0}^{\infty}$ be a decomposition of ω_1 into stationary sets. Then let $\mathcal{I}_{\vec{T}}$ be the ideal of all countable subsets of ω_1 which can be covered by finitely many of the T_n's. Clearly $\mathcal{I}_{\vec{T}}$ is $\aleph_1$-generated. It is also obvious that both of the alternatives of ($\mathrm{A}^{\star_c}$) must fail.

3. Random Aronszajn trees

The main results of this paper are contained in this Section, where measure algebraic names for locally countable trees are examined. An application to square sequences is given. We conclude with a discussion on larger trees, and we begin with a sufficiency for specialness.

Prerequisites and Notation: For a tree $\mathcal{T} = (T, \leq_T)$, let

$$\mathrm{pred}(t) = \{s \in T : s \leq_T t\}$$

for each $t \in T$. The **height** of a node, i.e. the order type of $(\mathrm{pred}(t) \setminus \{t\}, <_T)$, is denoted $\mathrm{ht}(t)$, and we write T_α for the α^{th} level of $\mathcal{T}$, i.e. $T_\alpha = \{t \in T : \mathrm{ht}(t) = \alpha\}$. For a set Γ of ordinals, $T \restriction \Gamma = \bigcup_{\alpha \in \Gamma} T_\alpha$. The **height** of the tree $\mathcal{T}$ is the minimum ordinal α such that $T_\alpha = \emptyset$. By a **subtree** of $\mathcal{T}$ we mean a tree of the form $(S, \leq_T)$ where $S \subseteq T$. Note that this disagrees with many authors' usage of the term (as a downwards closed subset). For a node $t \in T$, we write $\mathrm{imsucc}(t)$ for its set of **immediate successors**. And $\mathcal{T}$ has **unique limits** if nodes at limit levels are uniquely determined by their predecessors at lower levels, i.e. if $\lim_{\xi \to \mathrm{cf}(\delta)} \mathrm{ht}(t_\xi) = \delta$ for some limit δ, then $\{t_\xi : \xi < \mathrm{cf}(\delta)\}$ has at most one $\leq_T$-upper bound in the δ^{th} level of $\mathcal{T}$.

An ω_1**-tree** is a tree of height ω_1 with all levels countable. Recall that an **Aronszajn tree** is an ω_1-tree with no **cofinal branches**, i.e. a branch reaching all levels of the tree. And a tree of size $\aleph_1$ is called **special** if it has a countable decomposition into antichains.

We shall identify a tree $(T, \leq_T)$ with the collection $\{\mathrm{pred}(t) : t \in T\}$. Thus, for example, every ω_1-tree is locally countable (cf. page 6), i.e. the family $\{\mathrm{pred}(t) : t \in T\}$ has a countable trace on every countable subset of T.

LEMMA 3.1: *Let $(T, \leq_T)$ be a tree and let $S \subseteq T$ be a subtree. If S is orthogonal to $(T, \leq_T)$, then $(S, \leq_T)$ is of height at most ω.*

Proof: Let S be orthogonal to $(T, \leq_T)$, and suppose to the contrary that there is a $t \in S$ such that

$$\mathrm{ht}_{(S,\leq_T)}(t) = \omega. \tag{7}$$

This implies that $S \cap \mathrm{pred}(t)$ is infinite, contradicting the fact that $S \perp (T, \leq_T)$. ∎

Definition 3.2: By a **Cantor tree** we mean an uncountable tree $(T, \leq_T)$ of height $\omega + 1$ with unique limits such that $T \restriction \omega$ is countable.

It is known (e.g. [Tod00]), and easily proved, that an equivalent formulation is:

LEMMA 3.3: *A tree contains no Cantor subtrees iff it is locally countable.*

Since an Aronszajn tree is a tree of size and height ω_1 with neither a cofinal branch nor a Cantor subtree, one might consider strengthening the statement SH^+ (all Aronszajn trees are special) in this direction. However, by the following Theorem 3.4, this would be purely formal, i.e. SH^+ is equivalent the the statement: *every locally countable tree of size and height ω_1 either has a cofinal branch or is special.*

The following theorem is presumed to be part of the folklore.

THEOREM 3.4: *Let $\mathcal{T} = (T, \leq_T)$ be a tree of size and height at most ω_1. Then the following are equivalent:*

(a) *$\mathcal{T}$ does not contain a Cantor subtree.*

(b) *$\mathcal{T}$ is a subtree of an ω_1-tree $\mathcal{U}$ such that every cofinal branch through $\mathcal{U}$ contains an uncountable branch through $\mathcal{T}$.*

Proof: We prove the nontrivial implication. Let $\mathcal{T} = (T, \leq_T)$ be a tree of size and height at most ω_1, and fix a well-ordering $\lhd$ of its set of nodes T. Without loss of generality, assume that T_0 is countable (we could add a single minimum element to $\mathcal{T}$). We construct a tree $\mathcal{U} = (U, \leq_U)$, such that $\leq_U$ agrees with $\leq_T$ on $T \cap U$, and a function $f\colon U \setminus T \to \mathbb{Q}$, level by level by recursion on $\alpha < \omega_1$ so that

(i) f is strictly increasing on intervals of $U \setminus T$, i.e. if $[x, y]_{\mathcal{U}} \cap T = \emptyset$ then f is strictly increasing on $[x, y]_{\mathcal{U}}$, where $[x, y]_{\mathcal{U}} = \{u \in U : x \leq_U u \leq_U y\}$,

(ii) for all $x \in (U \restriction \alpha) \setminus T$,

(1) $\{y \in U_\alpha \setminus T : x \leq_U y, [x,y]_{\mathcal{U}} \subseteq U \setminus T\} \neq \emptyset$,

(2) $\inf\{f(y) : y \in U_\alpha \setminus T, x \leq_U y, [x,y]_{\mathcal{U}} \subseteq U \setminus T\} = f(x)$,

(iii) $U_0 = T_0$,

(iv) U_α is countable,

(v) for all $b \in \{\mathrm{pred}_{\mathcal{U}}(u) \cap T : u \in U_\alpha\}$, if b does have a $<_T$-upper bound in $T \setminus (U \restriction \alpha+1)$, then the $\lhd$-minimum such $<_T$-minimal upper bound is in $U_{\alpha+1}$,

(vi) $\{\mathrm{pred}_{\mathcal{T}}(t) \cap (U \restriction \alpha) : t \in T\} \subseteq \{\mathrm{pred}_{\mathcal{U}}(x) \cap T : x \in U_\alpha \setminus T\}$.

Let us verify that this is possible. Since there is no difficulty at successor levels, assume that α is a limit and $(U \restriction \alpha, \leq_U)$ and $f \restriction (U \restriction \alpha) \setminus T$ have been defined. For each member b of

$$B = \{b = \mathrm{pred}_{\mathcal{T}}(t) \cap (U \restriction \alpha) : t \in T, \sup_{s \in b} \mathrm{ht}_{\mathcal{U}}(s) = \alpha\}, \tag{8}$$

choose $z_b \notin (U \restriction \alpha) \cup T$ and extend $\leq_U$ so that $\mathrm{pred}_{\mathcal{U}}(z_b) \cap T = b$. For each $x \in (U \restriction \alpha) \setminus T$ and each rational $\varepsilon > 0$, by (ii) we can choose a sequence $x = y_0^{x,\varepsilon} \leq_U y_1^{x,\varepsilon} \leq_U \cdots$ with $[y_n^{x,\varepsilon}, y_{n+1}^{x,\varepsilon}]_{\mathcal{U}} \cap T = \emptyset$ and $\lim_{n\to\infty} \mathrm{ht}_{\mathcal{U}}(y_n^{x,\varepsilon}) = \alpha$ such that $f(y_n^{x,\varepsilon}) < f(x) + \varepsilon$ for all n. Then choose a new $z_{x,\varepsilon}$, and extend $\leq_U$ so that $y_n^{x,\varepsilon} \leq_U z_{x,\varepsilon}$ for all n. Letting

$$U_\alpha = \{z_b : b \in B\} \cup \{z_{x,1/(n+1)} : n \in \mathbb{N}, x \in (U \restriction \alpha) \setminus T\}, \tag{9}$$

conditions (1) and (vi) are satisfied, and hypothesis (a) ensures that condition (iv) is not violated. And letting $f(z_b)$ be arbitrary, and $f(z_{x,\varepsilon}) = \varepsilon$, conditions (i) and (2) are satisfied.

It easily follows from conditions (1),(v) and (vi) that $T \subseteq U$, and thus $\mathcal{T}$ is a subtree of $\mathcal{U}$. And (i) ensures that there is no cofinal branch through $\mathcal{U}$ contained in $U \setminus T$ modulo a countable set. ∎

Definition 3.5: A **labeling** of a set S by an ordinal θ is a bijection between S and θ. For a tree $\mathcal{T} = (T, \leq_T)$, a labeling $\{L_\alpha : \alpha \in \theta\}$ of a subset of T by θ is called **Aronszajn** if there is no cofinal $X \subseteq \theta$ such that $\{L_\alpha : \alpha \in X\}$ is a chain of $\mathcal{T}$.

Definition 3.6: A tree $\mathcal{T} = (T, \leq_T)$ is said to satisfy **property (T1)** if

(T1) For every Aronszajn labeling $\{L_\alpha : \alpha < \theta\}$ of a subtree of T by an ordinal θ of uncountable cofinality, there exists a closed uncountable $C \subseteq \theta$ such that $\{L_\alpha : \alpha \in C\}$ is orthogonal to $\mathcal{T}$.

LEMMA 3.7: *For every locally countable tree $\mathcal{T}$ of size and height ω_1 with no cofinal branches, if $\mathcal{T}$ satisfies property (T1) then $\mathcal{T}$ is special.*

Proof: Let $\mathcal{T} = (T, \leq_T)$ be such a tree. Then by Theorem 3.4, $\mathcal{T}$ is a subtree of an ω_1-tree $\mathcal{U} = (U, \leq_U)$ with no cofinal branches. It suffices to show that $\mathcal{U}$ is special. For each $\alpha < \omega_1$, let $\{x^n_\alpha\}_{n=0}^\infty$ enumerate U_α. By (T1) and Lemma 3.1, for every n, there is a club $C_n \subseteq \omega_1$ such that

$$\{x^n_\alpha : \alpha \in C_n\} \text{ is special.} \tag{10}$$

Let $C = \bigcap_{n=0}^\infty C_n$. For each n, write $\{x^n_\alpha : \alpha \in C\} = \bigcup_{i=0}^\infty A_{ni}$ where each A_{ni} is an antichain. Then $U \restriction C = \bigcup_{n=0}^\infty \bigcup_{i=0}^\infty A_{ni}$ shows that $U \restriction C$ is special.

We offer a proof of the well-known fact that if any ω_1-tree $(T, \leq_T)$ has a club $C \subseteq \omega_1$ such that $T \restriction C$ is special, then T is special. For each $\alpha \in C$, let

$$\beta(\alpha) = \min(C \setminus (\alpha + 1)).$$

And let $\{y^i_\alpha\}_{i=0}^\infty$ enumerate

$$\bigcup\{T_\xi : \xi \in [\alpha, \beta(\alpha))\}$$

for each $\alpha \in C$. Write $\bigcup_{\alpha \in C} T_\alpha = \bigcup_{j=0}^\infty A_j$ where each A_j is an antichain. For each $i, j \in \mathbb{N}$, define

$$B_{ij} = \{y^i_\alpha : \alpha \in C, y^i_\alpha \restriction \alpha \in A_j\}.$$

Since C is club, $\{[\alpha, \beta(\alpha)) : \alpha \in C\}$ is a (disjoint) partition of ω_1, and therefore $T = \bigcup_{i=0}^\infty \bigcup_{j=0}^\infty B_{ij}$. Hence it remains to show that every B_{ij} is an antichain. For fixed i and j, take $y \neq z$ in B_{ij}. Find $\alpha_y, \alpha_z \in C$ such that $y = y^i_{\alpha_y}$ and $z = y^i_{\alpha_z}$. Since $y \neq z$, $\alpha_y \neq \alpha_z$. And then from $y \restriction \alpha_y \in A_j$ and $z \restriction \alpha_z \in A_j$ we conclude that $y \perp_T z$ because A_j is an antichain. ∎

Let $\mathcal{R}$ be a measure algebra via $\mu\colon \mathcal{R} \to [0, 1]$. In what follows $\mathcal{T} = (T, \leq_T)$ is a given $\mathcal{R}$-name for a tree which contains no Cantor subtrees. Our goal is to prove that $\mathcal{T}$ satisfies property (T1) with probability one. Hence we can assume without loss of generality that the set T is in the ground model.

Definition 3.8: To an $\mathcal{R}$-name $L = \{L(\alpha) : \alpha < \check{\theta}\}$ for an Aronszajn labeling of a subset $\dot{S}$ of T, we associate an ideal $\mathcal{I}_L$ on θ consisting of all countable $\Omega \subseteq \theta$ such that

$$\|\{\alpha \in \Omega : L(\alpha) \leq_T t\} \text{ is finite}\| = 1 \quad \text{for all } t \in T.$$

Note that by going to a closed unbounded subset of θ we may assume that θ is regular.

LEMMA 3.9: *$\mathcal{I}_L$ is a P-ideal.*

Proof: Given $\{\Omega_n\}_{n=0}^{\infty} \subseteq \mathcal{I}_L$, set $\Omega_\omega = \bigcup_{n=0}^{\infty} \Omega_n$. By Lemma 3.3,

$$\mathcal{R} \Vdash \{\{L(\alpha) : \alpha \in \Omega_\omega\} \cap \operatorname{pred}(t) : t \in T\} \text{ is countable.} \tag{11}$$

And then by the ccc property of $\mathcal{R}$ we can find a countable $\Lambda \subseteq T$ such that

$$\mathcal{R} \Vdash \{\{L(\alpha) : \alpha \in \Omega_\omega\} \cap \operatorname{pred}(t) : t \in T\} \subseteq \{\{L(\alpha) : \alpha \in \Omega_\omega\} \cap \operatorname{pred}(t) : t \in \Lambda\}. \tag{12}$$

Let $\{t_i\}_{i=0}^{\infty}$ enumerate Λ.

For each i and n, there is a finite $\Gamma_{in} \subseteq \Omega_n$ such that

$$\mu\Bigg(\sum_{\alpha \in \Omega_n \setminus \Gamma_{in}} \|L(\alpha) \leq_T t_i\| \Bigg) \leq \frac{1}{n^2+1}. \tag{13}$$

Define

$$\Omega = \bigcup_{n=0}^{\infty} \Big(\Omega_n \setminus \bigcup_{i \leq n} \Gamma_{in} \Big).$$

It remains to show that $\Omega \in \mathcal{I}_L$. Suppose to the contrary that there are $t \in T$ and $a \in \mathcal{R}^+$ for which $a \Vdash \{\alpha \in \Omega : L(\alpha) \leq_T t\}$ is infinite. Then by (12), for some i and $b \leq a$,

$$b \Vdash \{\alpha \in \Omega : L(\alpha) \leq_T t_i\} \text{ is infinite.} \tag{14}$$

Pick m large enough so that $\sum_{n=m}^{\infty} \frac{1}{n^2+1} < \mu(b)$. Put

$$c = \sum \Big\{ \|L(\alpha) \leq_T t_i\| : \alpha \in \Omega \setminus \bigcup_{n<m} \Omega_n \Big\}.$$

Note that

$$\mu(c) \leq \sum_{n=m}^{\infty} \frac{1}{n^2+1} < \mu(b). \tag{15}$$

And thus $d = b - c \neq 0$. However, $d \Vdash \{\alpha \in \Omega : L(\alpha) \leq_T t_i\} \subseteq \bigcup_{n<m} \Omega_n$ contradicting (14). ∎

We also need to establish the following fact about $\mathcal{I}_L$:

LEMMA 3.10: *There is no cofinal subset of θ orthogonal to $\mathcal{I}_L$.*

The next Lemma is a reformulation of [Lav87, Lemma 2] in the language of ultraproducts, borrowed from [Tod96], and a generalization from $\aleph_1$ to arbitrary uncountable regular cardinals.

LEMMA 3.11 (Laver): *Suppose that $\mathcal{U}$ is a uniform ultrafilter on θ. Then given $X \in \theta^{[\theta]}$ and $f_\alpha \in \mathcal{R}^\theta$ $(\alpha \in X)$, if*

$$\lim_{\xi \to \mathcal{U}} \mu(f_\alpha(\xi)) \neq 0 \quad \textit{for all } \alpha \in X,$$

then there exists $a \in \mathcal{R}^+$ such that

$$a \Vdash \exists Y \in X^{[\theta]} \forall \alpha, \beta \in Y \exists \xi < \theta \; f_\alpha(\xi) \cdot f_\beta(\xi) \in \dot{\mathcal{G}}.$$

Proof: The proof in [Lav87] works if every occurrence of $\aleph_1$ is replaced with θ. ∎

For the time being we fix a subset X of θ of cardinality θ and a uniform ultrafilter $\mathcal{U}$ on θ containing X. For each $\alpha \in X$, define $g_\alpha \in \mathcal{R}^\theta$ by

$$g_\alpha(\xi) = \|L(\alpha) \leq_T L(\xi)\| \quad \text{for all } \xi < \theta.$$

CLAIM 3.12: $\lim_{\xi \to \mathcal{U}} \mu(g_\alpha(\xi)) = 0$ *for coboundedly many $\alpha \in X$.*

Proof: Suppose that the Claim is false. Then by Laver's Lemma there exists an $a \in \mathcal{R}^+$ and an $\mathcal{R}$-name $\dot{Y}$ for a subset of X of cardinality θ such that

$$a \Vdash \forall \alpha, \beta \in \dot{Y} \exists \xi \; g_\alpha(\xi) \cdot g_\beta(\xi) \in \dot{\mathcal{G}}. \tag{16}$$

But as $\mathcal{T}$ names a tree, this means that

$$a \Vdash \forall \alpha, \beta \in \dot{Y} \; L(\alpha) \leq_T L(\beta) \quad \text{or} \quad L(\beta) \leq_T L(\alpha), \tag{17}$$

contradicting our assumption that with probability one $\{L(\alpha)\}$ is an Aronszajn labeling of $\mathcal{T}$. ∎

By Claim 3.12 there is a $Y \in \mathcal{U}$ such that $\lim_{\xi \to \mathcal{U}} \mu(g_\alpha(\xi)) = 0$ for all $\alpha \in Y$. We choose $\{\alpha_n\}_{n=0}^\infty \subseteq Y$ by recursion on n so that

$$\|L(\alpha_n) \leq_T L(\alpha_i)\| = 0 \qquad \text{for all } i < n, \tag{18}$$

$$\mu\big(\|L(\alpha_i) \leq_T L(\alpha_n)\|\big) \leq 1/(n^3+1) \qquad \text{for all } i < n. \tag{19}$$

To see that this is possible, assume that $\{\alpha_i\}_{i<n}$ has been defined. Since

$$\mathcal{R} \Vdash |\operatorname{pred}(t) \cap \dot{S}| < \theta \quad \text{for all } t \in T, \tag{20}$$

and by the ccc property of $\mathcal{R}$, there exists a set $\Lambda \subseteq \theta$ of cardinality less than θ such that

$$\mathcal{R} \Vdash \bigcup_{i<n} \operatorname{pred}(L(\alpha_i)) \cap \dot{S} \subseteq \{L(\alpha) : \alpha \in \Lambda\}. \tag{21}$$

And by our choice of Y, for each $i < n$ there is a $Z_i \in \mathcal{U}$ such that

$$\mu(g_{\alpha_i}(\xi)) \leq 1/(n^3+1) \quad \text{for all } \xi \in Z_i. \tag{22}$$

Then any $\alpha_n \in Y \cap Z_0 \cap \cdots \cap Z_{n-1} \setminus \Lambda$ will work.

CLAIM 3.13: $\{\alpha_n : n \in \mathbb{N}\} \in \mathcal{I}_L$.

Proof: Let $a \in \mathcal{R}^+$ be given. Find m large enough so that

$$\sum_{n=m}^{\infty} \frac{n-m}{n^3+1} < \mu(a). \tag{23}$$

Now observe that the probability that $\{L(\alpha_n) : n \geq m\}$ is not an antichain is given by

$$b = \sum_{n=m}^{\infty} \sum_{\substack{i=m \\ i \neq n}}^{\infty} \|L(\alpha_i) \leq_T L(\alpha_n)\|. \tag{24}$$

But

$$\begin{aligned} \mu(b) &\leq \sum_{n=m}^{\infty} \sum_{\substack{i=m \\ i \neq n}}^{\infty} \mu\big(\|L(\alpha_i) \leq_T L(\alpha_n)\|\big) \\ &= \sum_{n=m}^{\infty} \sum_{i=m}^{n-1} \mu\big(\|L(\alpha_i) \leq_T L(\alpha_n)\|\big) && \text{by (18)} \\ &\leq \sum_{n=m}^{\infty} \frac{n-m}{n^3+1} < \mu(a) && \text{by (19).} \end{aligned}$$

Thus $c = a - b \neq 0$ and furthermore $c \Vdash \{L(\alpha_n) : n \geq m\}$ is an antichain. We have now proved that $\{L(\alpha_n) : n \in \mathbb{N}\}$ is the union of an antichain and a finite set with probability one. This suffices. ∎

We have now completed a proof of Lemma 3.10.

THEOREM 3.14: *For a cardinal κ, $(\star_c)$ for $\theta \leq \kappa$ implies that every $\mathcal{R}$-name for a tree of size κ containing no Cantor subtrees satisfies property (T1) with probability one.*

Proof: Suppose that $\mathcal{T} = (T, \leq_T)$ is a given $\mathcal{R}$-name for a tree of size κ which contains no Cantor subtrees. Let L be a given $\mathcal{R}$-name for an Aronszajn labeling of a subset of T by some uncountable ordinal θ. We need only concern ourselves with regular θ, in which case $\theta \leq \kappa$. Then Lemma 3.10 implies that the second alternative of $(\star_c)$ must fail, and therefore there is a closed uncountable $C \subseteq \theta$ such that $C^{[\aleph]} \subseteq \mathcal{I}_L$. However, from the ccc property of $\mathcal{R}$ it follows that

$$\|\{L(\alpha) : \alpha \in C\} \text{ is orthogonal to } \mathcal{T}\| = 1, \tag{25}$$

proving that $\mathcal{T}$ satisfies property (T1) with probability one. ∎

Corollary 3.15: *$(\star_c)$ restricted to ω_1 implies* RSH$^+$.

Proof: By Lemma 3.7 and Theorem 3.14. ∎

Corollary 3.16: *Conjecture 1 implies that* RSH$^+$ *is consistent with* CH.

Theorem 3.17: *$(*)$ restricted to ω_1 implies* RSH.

Proof: Suppose that $\mathcal{T} = (T, \leq_T)$ is an $\mathcal{R}$-name for Aronszajn tree on ω_1. We consider the ideal $\mathcal{I}_T$ (i.e. the labeling is the identity map), and by a similar argument to the proof of Theorem 3.14 we obtain an uncountable $X \subseteq \omega_1$ such that $X^{[\aleph]} \subseteq \mathcal{I}_T$, and thus is orthogonal to the tree with probability one. Then by Lemma 3.1, $(X, \leq_T)$ is special with probability one, and in particular, $\mathcal{T}$ has an uncountable antichain with probability one. ∎

Corollary 3.18: RSH *is consistent with* CH.

Proof: Theorems 2.1 and 3.17. ∎

3.1. Square sequences. Recall that for a given ordinal θ (typically regular and uncountable), a **square sequence on** θ is a sequence of the form C_α $(\alpha < \theta)$ where

(a) C_α is a closed unbounded subset of α when α is a limit, and $C_{\alpha+1} = \{\alpha\}$,
(b) $C_\alpha = C_\beta \cap \alpha$ whenever α is a limit point of C_β,
(c) for every closed unbounded $C \subseteq \theta$, there is limit point α of C such that $C \cap \alpha \neq C_\alpha$.

The statement *there exists a square sequence on* θ is denoted by $\square(\theta)$. In [Tod84a], a *partial square sequence on* θ is considered, that is a set $\Gamma \subseteq \theta$ of limit ordinals containing $\{\alpha < \theta : \mathrm{cf}(\alpha) = \aleph_1\}$ and a sequence $\langle C_\alpha : \alpha \in \Gamma\rangle$ satisfying (a)–(c) for every $\alpha \in \Gamma$, and also

(d) $\alpha \in \Gamma$ whenever α is a limit point of C_β.

For an infinite cardinal κ, a $\square_\kappa$**-sequence** is sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ satisfying (a) and (b), and with condition (c) strengthened to

(e) $\operatorname{otp}(C_\alpha) \leq \kappa$ for all $\alpha < \kappa^+$.

The statement *there exists a $\square_\kappa$-sequence* is denoted by $\square_\kappa$. Thus $\square_\kappa$ implies $\square(\kappa^+)$.

It is shown in [Tod84a] that:

THEOREM 3.19 (Todorčević): PFA *implies that there is no partial square sequence on any ordinal of cofinality strictly greater than $\aleph_1$. In particular,* PFA *implies $\neg\square_\kappa$ for every uncountable cardinal κ.*

There is a natural correspondence between a (partial) square sequence on θ and a tree ordering $<^2$ on Γ given by

(iii) $$\alpha <^2 \beta \quad \text{iff } \alpha \text{ is a limit point of } C_\beta$$

for all $\alpha, \beta \in \Gamma$. Observe that condition (c) can be reformulated as

(f) the tree $(\Gamma, \leq^2)$ has no branch which is cofinal as a subset of θ.

To see this, suppose that $B \subseteq \Gamma$ is a cofinal set which is a branch. Notice that B consists entirely of limit ordinals. Define a club subset of θ by

$$C = \bigcup \{C_\beta : \beta \in B\}.$$

Fix a limit $\alpha \in C$. Set $\bar{\beta} = \min(B \setminus (\alpha+1))$. Since $\alpha \in C_\gamma$ for some $\gamma \in B \setminus (\alpha+1)$, it follows from (b) that $\alpha \in C_{\bar{\beta}}$, and then it follows from (b) that

(26) $$C_\beta \cap \alpha = C_\alpha \quad \text{for all } \beta \in B \setminus (\alpha + 1).$$

Now since $C_\beta \cap \alpha = C_\beta = C_{\bar{\beta}} \cap \beta \subseteq C_\alpha$ for all $\beta \in B \cap (\alpha + 1)$, we have $C \cap \alpha = C_\alpha$. Therefore C witnesses that condition (c) fails. Conversely, if $C \subseteq \theta$ is a club witnessing the failure of (c), then the set of limit points of C is a branch in $(\Gamma, \leq^2)$.

LEMMA 3.20: $(\Gamma, \leq^2)$ *contains no Cantor subtrees.*

Proof: By Lemma 3.3, we need to show that $\{\operatorname{pred}(\beta) : \beta \in \Gamma\}$ is locally countable. Fix a countably infinite $\Omega \subseteq \theta$. We may assume that Ω consists entirely of limit ordinals. Then by conditions (a) and (b),

(27) $$\{\Omega \cap \operatorname{pred}(\beta) : \beta \in \Gamma\} = \{\Omega \cap C_\alpha : \alpha \in \overline{\Omega}\},$$

where $\overline{\Omega}$ denotes the closure of Ω. This concludes the proof. ∎

Now we can see the influence of $(\star_c)$ on random square sequences—and in particular on square sequences:

THEOREM 3.21: *$(\star_c)$ restricted to $\theta \leq \kappa$ implies that there is no partial square sequence on any ordinal $\theta \leq \kappa$ of cofinality strictly greater than $\aleph_1$, in any extension by a measure algebra.*

Proof: Suppose towards a contradiction that $\dot{C}_\alpha$ ($\alpha \in \dot{\Gamma}$) is an $\mathcal{R}$-name for a partial square sequence on θ. By Lemma 3.20, with probability one $(\dot{\Gamma}, \leq^2)$ contains no Cantor subtrees. Therefore, by Theorem 3.14, $(\dot{\Gamma}, \leq^2)$ satisfies property (T1) with probability one. Let $\dot{\gamma}_\alpha$ ($\alpha < \theta$) be an $\mathcal{R}$-name for the strictly increasing enumeration of $\dot{\Gamma}$. Then condition (f) says that $\{\dot{\gamma}_\alpha\}$ is an Aronszajn labeling with probability one. Thus there is a closed uncountable $C \subseteq \theta$ such that $\{\dot{\gamma}_\alpha : \alpha \in C\}$ is orthogonal to $(\dot{\Gamma}, \leq^2)$ with probability one. Let D be the initial segment of C of length ω_1. Then there is an $a \in \mathcal{R}^+$ and a $\beta < \theta$ such that

$$a \Vdash \beta = \sup\{\dot{\gamma}_\alpha : \alpha \in D\}. \tag{28}$$

Since $a \Vdash \beta \in \dot{\Gamma}$, it follows from (d) that

$$a \Vdash \dot{C}_\beta \cap \{\dot{\gamma}_\alpha : \alpha \in D\} \text{ is a closed unbounded subset of } \beta, \tag{29}$$

and in particular a forces that $\{\dot{\gamma}_\alpha : \alpha \in D\}$ is not orthogonal to $(\dot{\Gamma}, \leq^2)$. This contradiction concludes the proof. ∎

Remark 3.22: This together with Theorem 4.7 gives a proof of Theorem 3.19.

3.2. LARGER TREES. We consider trees which are large in the sense that they contain a Cantor subtree. We have just seen that $(\star_c)$ implies that every locally countable tree of size $\aleph_1$ either has an uncountable branch, or can be decomposed into countably many antichains. Recall that $\mathrm{MA}_{\aleph_1}$ implies that every tree of size $\aleph_1$ either has an uncountable branch or a countable decomposition into antichains (see [BMR70]). On the other hand, under CH, there exists a tree of size $\aleph_1$ which neither has an uncountable branch, nor can be decomposed into countably many antichains; for example, the tree $(\sigma\mathbb{Q}, \sqsubseteq)$, where $\sigma\mathbb{Q}$ is the set of all well-ordered subsets of $(\mathbb{Q}, \leq)$ and '$\sqsubseteq$' denotes the end-extension ordering, does not satisfy either of these two alternatives (see [Tod84b, §9]).

Furthermore, while forcing with a measure algebra $\mathcal{R}$ of weight at least $\aleph_2$ violates CH, in $V^{\mathcal{R}}$, the tree $(\sigma\mathbb{Q}, \sqsubseteq)^V$ has size $\aleph_1$ yet it fails to satisfy either of the two alternatives mentioned above. This fact is a consequence of Lemma 3.24 below.

Definition 3.23: A poset $(\mathcal{P}, \leq)$ is said to satisfy the **σ-finite chain condition** if there is a countable decomposition of $\mathcal{P}$ into sets which contain no infinite antichains of $\mathcal{P}$.

Every measure algebra $(\mathcal{R}, \mu)$ has the σ-finite chain condition because we can write $\mathcal{R}^+ = \bigcup_{n=0}^{\infty} \mathcal{R}_n$ where $\mathcal{R}_n = \{a \in \mathcal{R}^+ : \mu(a) \geq 1/(n+1)\}$.

LEMMA 3.24 (Todorčević): *Posets which satisfy the σ-finite chain condition do not specialize nonspecial trees, nor do they add uncountable chains to tree orderings which do not already have them.*

Proof: In [Tod85, Lemma 8] it is shown that no poset with the σ-finite chain condition specializes a nonspecial tree. It is also well-known that no poset satisfying property K adds an uncountable branch to a tree ordering (see [KT79]). And it follows from the Dushnik–Miller relation $\omega_1 \to (\omega_1, \omega)$ that every poset with the σ-finite chain condition has the property K. ∎

4. The consistency of $(\star_c)$

Although we are primarily interested in the consistency of $(\star_c)$ with CH, we take this opportunity to show that it is a consequence of PFA. Given a P-ideal $\mathcal{I}$ on some ordinal θ of uncountable cofinality, for which there is no stationary subset of θ orthogonal to $\mathcal{I}$, we need a proper poset $\mathcal{P}_{\mathcal{I}}$ that forces a closed uncountable subset C of θ such that $C^{[\aleph_0]} \subseteq \mathcal{I}$. Our poset is based on the corresponding posets in [AT97] and [Tod00]. The essential difference is that our working condition is now closed.

Definition 4.1: For an ideal $\mathcal{I}$ on θ, let $\mathcal{P}$ $(= \mathcal{P}_{\mathcal{I}})$ be the poset consisting of all pairs $p = (x_p, \mathcal{X}_p)$ where

(a) x_p is a countable closed subset of θ,

(b) $\mathcal{X}_p$ is a countable set of *cofinal* subsets of $\mathcal{I}$, called **promises**, where $\mathcal{I}$ is ordered by $\subseteq^*$,

ordered by $(x_q, \mathcal{X}_q) \leq (x_p, \mathcal{X}_p)$ iff

(c) $x_q \sqsupseteq x_p$,

(d) $\mathcal{X}_q \supseteq \mathcal{X}_p$,

(e) for every $X \in \mathcal{X}_p$, the set $\{\Omega \in X : x_q \setminus x_p \subseteq \Omega\}$ is cofinal in $\mathcal{I}$ and belongs to $\mathcal{X}_q$.

LEMMA 4.2: *If $\mathcal{I}$ is a P-ideal then the ideal of noncofinal subsets of $\mathcal{I}$ is a σ-ideal.*

Proof: Let $\mathcal{X}$ be a countable collection of noncofinal subsets of $\mathcal{I}$. Then for each $X \in \mathcal{X}$, there exists $\Omega_X \in \mathcal{I}$ such that

$$\Omega_X \setminus \Omega \text{ is infinite} \quad \text{for all } \Omega \in X. \tag{30}$$

Since $\mathcal{I}$ is a P-ideal, there exists $\Omega_\omega \in \mathcal{I}$ such that $\Omega_X \subseteq^* \Omega_\omega$ for all $X \in \mathcal{X}$. Then Ω_ω witnesses that $\bigcup \mathcal{X}$ is not cofinal in $\mathcal{I}$. ∎

In following Lemmas 4.3, 4.4, 4.5 and 4.6, we are assuming that $\mathcal{I}$ is a P-ideal on some ordinal θ of uncountable cofinality with no stationary orthogonal set, and κ is always some sufficiently large regular cardinal, i.e. $\kappa \geq (2^{|\theta|^{\aleph_0}})^+$.

Lemma 4.3: *For every countable $M \prec H_\kappa$ containing $\mathcal{I}$, for every cofinal $X \subseteq \mathcal{I}$ in M,*

$$\{\Omega \in X : \sup(\theta \cap M) \in \Omega\} \text{ is cofinal in } \mathcal{I}.$$

Proof: Let A be the set of all $\alpha < \theta$ such that

$$Y_\alpha = \{\Omega \in X : \alpha \in \Omega\} \text{ is not cofinal in } \mathcal{I}. \tag{31}$$

Note that A is orthogonal to $\mathcal{I}$, because for every countably infinite $\Lambda \subseteq A$,

$$\{\Omega \in X : \Lambda \subseteq^* \Omega\} \subseteq \bigcup_{\alpha \in \Lambda} Y_\alpha, \tag{32}$$

which by Lemma 4.2 is not cofinal in $\mathcal{I}$; and this implies that $\Lambda \notin \mathcal{I}$. By our assumption on $\mathcal{I}$, A is not stationary, and therefore $\sup(\theta \cap M) \notin A$, as $A \in M$. ∎

Lemma 4.4: *For every $p \in \mathcal{P}$ and $\eta < \theta$, there exists $q \leq p$ such that $\min(x_q \setminus x_p) \geq \eta$.*

Proof: Let p and η be given. Pick a countable $M \prec H_\kappa$ containing p, η and $\mathcal{I}$. Put $\delta = \sup(\theta \cap M)$. By Lemma 4.3,

$$\{\Omega \in X : \delta \in \Omega\} \text{ is cofinal in } \mathcal{I} \tag{33}$$

for every $X \in \mathcal{X}_p$. Therefore there is a q extending p with $x_q = x_p \cup \{\delta\}$. ∎

Lemma 4.5: *For a countable $M \prec H_\kappa$ containing $\mathcal{I}$, suppose Ω is a set such that $\Lambda \subseteq^* \Omega$ for all $\Lambda \in \mathcal{I} \cap M$. Then for every $p_0 \in \mathcal{P} \cap M$ and every dense $\mathcal{D} \subseteq \mathcal{P}$ in M, there is an extension $q \leq p_0$ in $\mathcal{D} \cap M$ such that $x_q \setminus x_{p_0} \subseteq \Omega$.*

Proof: Suppose not. Let $\mathcal{J}$ be the set of all $\Lambda \in \mathcal{I}$ for which there exists $\Omega_\Lambda \in \mathcal{I}$ such that $\Lambda \subseteq^* \Omega_\Lambda$ and there is no $q \leq p_0$ in $\mathcal{D}$ such that $x_q \setminus x_{p_0} \subseteq \Omega_\Lambda$.

Notice that for every $\Lambda \in \mathcal{I} \cap M$, $\Omega_\Lambda = \Lambda \cap \Omega \in \mathcal{I} \cap M$ and witnesses that $M \models \Lambda \in \mathcal{J}$. Therefore, as $\mathcal{J} \in M$, $\mathcal{J} = \mathcal{I}$. Since $X = \{\Omega_\Lambda : \Lambda \in \mathcal{I}\}$ is cofinal in $\mathcal{I}$, we can define an extension p_1 of p_0 by

$$p_1 = (x_{p_0}, \mathcal{X}_{p_0} \cup \{X\}).$$

Then we can find $p_2 \leq p_1$ in $\mathcal{D}$. But

$$X_1 = \{\overline{\Omega} \in X : x_{p_2} \setminus x_{p_0} \subseteq \overline{\Omega}\} \tag{34}$$

is cofinal in $\mathcal{I}$, and in particular nonempty. Take $\overline{\Omega} \in X_1$. Then $\overline{\Omega} = \Omega_\Lambda$ for some $\Lambda \in \mathcal{I}$, and thus $x_{p_2} \setminus x_{p_0} \subseteq \Omega_\Lambda$ contrary to the fact that Ω_Λ witnesses that $\Lambda \in \mathcal{J}$. ∎

A poset $\mathcal{Q}$ is **completely proper** if for every countable $M \prec H_\lambda$ containing $\mathcal{Q}$ for λ a sufficiently large regular cardinal, every $p \in \mathcal{P} \cap M$ has a **completely** $(M, \mathcal{Q})$**-generic** extension q, i.e. for all dense $\mathcal{D} \subseteq \mathcal{Q}$ in M, q extends some member of $\mathcal{D} \cap M$. Note that a poset is completely proper iff it is proper and does not add reals.

LEMMA 4.6: *$\mathcal{P}$ is completely proper.*

Proof: Let $M \prec H_\kappa$ be a countable elementary submodel containing $\mathcal{P}$, and let $\{\mathcal{D}_n\}_{n=0}^\infty$ enumerate all of the dense subsets of $\mathcal{P}$ in M. Fix a set $\Omega_M \in \mathcal{I}$ such that $\Omega \subseteq^* \Omega_M$ for all $\Omega \in \mathcal{I} \cap M$. Given $p_0 \in \mathcal{P} \cap M$, we choose a decreasing sequence of conditions $p_0 \geq p_1 \geq \cdots$ such that $p_{n+1} \in \mathcal{D}_n \cap M$ for all n, and so that when we let y be the union of the x_{p_n}'s

$$\mathcal{Z}(X, n) = \{\Omega \in X : (y \cup \{\sup(\theta \cap M)\}) \setminus x_{p_n} \subseteq \Omega\} \text{ is cofinal in } \mathcal{I} \quad \text{for all } n, \text{ for all } X \in \mathcal{X}_{p_n}. \tag{35}$$

If we succeed in doing this, we will let

$$\mathcal{X}_{p_\omega} = \bigcup_{n=0}^{\infty} (\mathcal{X}_{p_n} \cup \{\mathcal{Z}(X, n) : X \in \mathcal{X}_{p_n}\}),$$

and then put

$$p_\omega = (y \cup \{\sup(\theta \cap M)\}, \mathcal{X}_{p_\omega}).$$

Since, by Lemma 4.4, $y \cup \{\sup(\theta \cap M)\}$ will be closed in θ, p_ω will be a member of $\mathcal{P}$. And it will follow from (35) that $p_\omega \leq p_n$ for all n, completing the proof.

In order to do this we shall need a bookkeeping device, say mappings $\varphi_0, \varphi_1 \colon \omega \to \omega$ where $\varphi_0(n) \leq n$ for all n, and $n \mapsto (\varphi_0(n), \varphi_1(n))$ defines a surjection. For each n, having chosen p_n, let $\{X_{ni}\}_{i=0}^{\infty}$ be an enumeration of $\mathcal{X}_{p_n}$. The idea is that the promise X_{mi} is taken care of when choosing p_{n+1} where $(\varphi_0(n), \varphi_1(n)) = (m, i)$.

Given n, assume that p_n has been defined. Let $(m, i) = (\varphi_0(n), \varphi_1(n))$. Put

$$Y = \{\Omega \in X_{mi} : x_{p_n} \setminus x_{p_m} \subseteq \Omega\},$$

and note that Y is cofinal in $\mathcal{I}$ because $p_n \leq p_m$. Then by Lemma 4.3 (note that $\mathcal{I} \in M$),

$$Y_0 = \{\Omega \in Y : \sup(\theta \cap M) \in \Omega\}$$

is cofinal in $\mathcal{I}$. Since $\{\Omega \in Y_0 : \Omega_M \subseteq^* \Omega\}$ is cofinal in $\mathcal{I}$, by Lemma 4.2 there is a finite $F_n \subseteq \Omega_M$ such that

$$\{\Omega \in Y_0 : \Omega_M \setminus F_n \subseteq \Omega\} \text{ is cofinal in } \mathcal{I}. \tag{36}$$

Now $\mathcal{Z}(X_{mi}, m)$ will be cofinal in $\mathcal{I}$ as long as

$$x_{p_{j+1}} \setminus x_{p_j} \subseteq \Omega_M \setminus F_n \quad \text{for all } j \geq n. \tag{37}$$

To ensure (37) it suffices to find $p_{n+1} \leq p_n$ such that

$$p_{n+1} \in \mathcal{D}_n \cap M, \tag{38}$$

$$x_{p_{n+1}} \setminus x_{p_n} \subseteq \Omega_M \setminus (F_0 \cup \cdots \cup F_n). \tag{39}$$

And for this we have Lemma 4.5. ∎

THEOREM 4.7: PFA *implies* $(\star_c)$.

Proof: This follows immediately from the fact that

$$\mathcal{D}_\xi = \{p \in \mathcal{P} : \mathrm{otp}(x_p) > \xi\} \tag{40}$$

is dense for every countable ordinal ξ. This in turn is so because if $\mathcal{P} \in M \prec H_\kappa$ then for any $(M, \mathcal{P})$-generic filter $\mathcal{G}$,

$$M[\mathcal{G}] \models \bigcup_{p \in \mathcal{G}} x_p \text{ is club in } \theta. \tag{41}$$

∎

5. Random Kurepa trees

Definition 5.1: A tree $(T, \leq_T)$ is called **essentially special** if there is a map $f\colon T \to \omega$ such that

$$s \leq_T t, u \quad \text{and} \quad f(s) = f(t) = f(u)$$

implies

$$t \text{ and } u \text{ are } \leq_T \text{-comparable}$$

for all $s, t, u \in T$. Note that $(T, \leq_T)$ is essentially special iff it can be decomposed into countably many antichains of $\leq_T$-chains; and that on a tree of size $\aleph_1$ with no branches of length ω_1, essential specialness and specialness coincide.

The notion of essential specialness is due to Baumgartner [Bau83] who used it, with the following Lemma (for the case of ω_1-trees), for the purpose of proving that PFA implies the nonexistence of Kurepa trees (see [Dev83]).

LEMMA 5.2 (Baumgartner): *If $(T, \leq_T)$ is an essentially special tree with height of uncountable cofinality, then $(T, \leq_T)$ has at most $|T|$ cofinal branches.*

Proof: Suppose that $f\colon T \to \omega$ is an essentially specializing map. For each cofinal branch B of $(T, \leq_T)$ there is an $t_B \in B$ such that

$$\{s \in B : f(s) = f(t_B)\} \text{ is cofinal in } B. \tag{42}$$

Now for any two cofinal branches B and C, note that if $t_B = t_C$ then $B = C$. ∎

The following Theorem which randomizes the corresponding Theorem of Baumgartner in [Dev83] appears in some unpublished notes of Baumgartner [Bau]. We will use the results of this section to provide a proof.

THEOREM 5.3 (Baumgartner): PFA *implies that every tree of size and height ω_1 in every forcing extension by a measure algebra is essentially special. In particular,* PFA *implies that every tree of size and height ω_1 has at most $\aleph_1$ cofinal branches (and in particular there are no Kurepa trees) in any forcing extension by a measure algebra.*

In this Section we discuss the statement that all ω_1-trees are essentially special in any forcing extension by a measure algebra (denoted $\neg\,\mathrm{RKH}^-$ in the Introduction). Since this does not appear to be a consequence of $(\star_c)$, we make a separate conjecture.

CONJECTURE 2: $\neg$ RKH$^-$ *is consistent with* CH *relative to the existence of an inaccessible cardinal.*

All of the necessary ingredients for a proof are provided, with the exception of the iteration theory.

As was the case with SH$^+$, the statement that every ω_1-tree is essentially special (denoted $\neg$ KH$^-$ in the Introduction), has a purely formal strengthening. I.e. by Theorem 3.4, $\neg$ KH$^-$ is equivalent to: *all locally countable trees of size and height* ω_1 *are essentially special.* One should note that this is the best approximation to Theorem 5.3 which is compatible with CH, since, for example, the complete binary tree of height ω_1 has cardinality $\aleph_1$ under CH.

LEMMA 5.4: *If* $\mathcal{T} = (T, \leq)$ *is a tree of size and height* ω_1 *which has at most* $\aleph_1$ *cofinal branches, then there is a subtree* $\mathcal{T}^*$ *of* $\mathcal{T}$ *with the property that*

(a) $\mathcal{T}^*$ *has no branches of length* ω_1,
(b) $\mathcal{T}$ *is essentially special iff* $\mathcal{T}^*$ *is a countable union of antichains,*
(c) *if* $\mathcal{T}^*$ *is a countable union of antichains in some forcing extension, then* $\mathcal{T}$ *is essentially special in that extension.*

Proof: Let $\mathcal{T} = (T, \leq)$ be as in the hypothesis of the Lemma, and let $U \subseteq T$ be the set of all nodes which are not in any cofinal branch of $\mathcal{T}$. Let $\{b_\alpha : \alpha < \lambda\}$ $(\lambda \leq \omega_1)$ be a 1–1 enumeration of all the cofinal branches of $\mathcal{T}$. For each $\alpha < \lambda$, define $x_\alpha \in b_\alpha$ as the $\leq_T$-least element of

$$b_\alpha \setminus \bigcup_{\xi<\alpha} b_\xi.$$

Define a subtree $\mathcal{T}^* = (T^*, \leq_T)$ by

$$T^* = \{x_\alpha : \alpha < \lambda\} \cup U.$$

For each α, let $c_\alpha = \{y \in b_\alpha : x_\alpha \leq_T y\}$. If in some forcing extension we can write $T^* = \bigcup_{n=0}^{\infty} A_n$ where each A_n is an antichain, then observe that $\mathcal{C}_n = \{c_\alpha : x_\alpha \in A_n\}$ is an antichain of chains for all n, and that $\{\mathcal{C}_n : n < \omega\}$ is a decomposition of $T \setminus U$, and that U is essentially special, proving (c). The other implication in (b) follows from the fact that $T^* \cap C$ is countable for any chain C of $\mathcal{T}$. ∎

LEMMA 5.5: *Suppose that* $\mathcal{T} = (T, \leq_T)$ *is an* $\mathcal{R}$*-name for an* ω_1*-tree with at most* $\aleph_1$ *cofinal branches for some measure algebra* $\mathcal{R}$*. Then there is an iteration* $\mathcal{Q}(\mathcal{T})$*, of completely proper posets, forcing that*

$$\|\mathcal{T} \textit{ is essentially special}\| = 1.$$

Proof: Given $\mathcal{T}$, let $\mathcal{T}^*$ be an $\mathcal{R}$-name for the tree from Lemma 5.4. Put

$$a = \|\,\mathrm{ht}(\mathcal{T}^*) = \omega_1\|. \tag{43}$$

By Lemma 5.4 (b), $-a \leq \|\mathcal{T}$ is essentially special$\|$. Hence, by going to the subalgebra $\mathcal{R}_a = \{b \in \mathcal{R} : b \leq a\}$ we can assume that $\mathcal{T}^*$ is of height ω_1 with probability one. Since $\mathcal{T}$ contains no Cantor subtrees, neither does $\mathcal{T}^*$, and thus by Theorem 3.4, with probability one, $\mathcal{T}^*$ is a subtree of an ω_1-tree $\mathcal{U} = (U, \leq_U)$ with no cofinal branches. For each $\alpha < \omega_1$, let $\{\dot{x}^n_\alpha\}_{n=0}^\infty$ be an $\mathcal{R}$-name for an enumeration of U_α. For each n, let L_n be the $\mathcal{R}$-name for the labeling of a subtree of $\mathcal{U}$ by ω_1 given by $\|L_n(\alpha) = \dot{x}^n_\alpha\| = 1$. From the proof of Lemma 3.7, to specialize $\mathcal{T}^*$ it suffices to force for each n a closed unbounded $C_n \subseteq \omega_1$ such that $\|\{L_n(\alpha) : \alpha \in C_n\} \in \mathcal{U}^\perp\| = 1$. And this is precisely what the poset $\mathcal{P}_\mathcal{I}$ does, where $\mathcal{I} = \mathcal{I}_{L_n}$ is the P-ideal from Definition 3.8 which has no stationary subset of ω_1 orthogonal to it (Lemma 3.10). Thus the poset

$$\mathcal{Q}(\mathcal{T}) = \prod_{n<\omega} \mathcal{P}_{\mathcal{I}_{L_n}}, \tag{44}$$

specializes $\mathcal{T}^*$, and hence by Lemma 5.4 (c) essentially specializes $\mathcal{T}$. Furthermore, by Lemma 4.6 each of the posets in the product is completely proper. ∎

Remark 5.6: The posets $\mathcal{P}_{\mathcal{I}_{L_n}}$ are also complete for some simple completeness system, cf. [She82] (we did not prove this here). Thus $\mathcal{Q}(\mathcal{T})$ is in fact completely proper, because iterations of length shorter than ω^2 of such posets do not add reals.

It is a well-known theorem of Silver ([Sil71]) that posets which are σ-closed do not add new cofinal branches to ω_1-trees. In [Dev83] the (necessary) additional axiom that CH fails, is used to generalize the conclusion of Silver's Theorem to all trees of size and height ω_1. In [Bau] this is generalized further as follows:

Lemma 5.7 (Baumgartner): *$\mathfrak{c} > \aleph_1$ implies that for any measure algebra $\mathcal{R}$, no σ-closed poset adds an $\mathcal{R}$-name for a new uncountable branch of an $\mathcal{R}$-name for a tree of size and height ω_1.*

Proof: See [Bau] and also Remark 5.10. ∎

We modify the proof in [Bau] to obtain a strengthening of Silver's result.

LEMMA 5.8: *If $\mathcal{Q}$ is a poset with the ccc and $\dot{\mathcal{T}}$ is a $\mathcal{Q}$-name for an ω_1-tree, then no σ-closed poset adds a $\mathcal{Q}$-name for a new cofinal branch of $\dot{\mathcal{T}}$.*

Proof: Suppose, without loss of generality, that we are given a Boolean algebra $\mathcal{B}$ with the ccc and a $\mathcal{B}$-name $\dot{\mathcal{T}} = (\dot{T}, \leq_{\dot{T}})$ for an ω_1-tree. We may assume that $\dot{\mathcal{T}}$ names a tree on ω_1, and moreover we may assume that

(45) $$\mathcal{B} \Vdash \dot{T}_\alpha = [\omega \cdot \alpha, \omega \cdot \alpha + \omega) \quad \text{for all } \alpha < \omega_1.$$

Define a $\mathcal{B}$-name $\dot{B}$ by

$$\mathcal{B} \Vdash \dot{B} = \{y \subseteq \omega_1 : y \text{ is a cofinal branch of } \dot{\mathcal{T}}\}.$$

For a $\mathcal{Q}$-name $\dot{y}$ for a subset of ω_1, we define $f_\alpha(\dot{y}) \in \mathcal{B}^\omega$ $(\alpha < \omega_1)$ by

$$f_\alpha(\dot{y})(n) = \|\omega \cdot \alpha + n \in \dot{y}\|_{\mathcal{B}} \quad \text{for all } n < \omega.$$

Let $\mathcal{P}$ be a given σ-closed poset, and note that

(46) $$\mathcal{P} \Vdash \mathcal{B} \text{ has the ccc.}$$

Now suppose towards a contradiction that the conclusion of the Lemma fails. Then there exists $(p_0, b_0) \in \mathcal{P} \times \mathcal{B}$ and a $\mathcal{P}$-name $\dot{\dot{x}}$ for a $\mathcal{B}$-name for a cofinal branch of $\dot{\mathcal{T}}$ where

(47) $$(p_0, b_0) \Vdash \dot{\dot{x}} \notin \check{\dot{B}}.$$

We assume for notational convenience that $b_0 = 1$.

SUBLEMMA 5.9: *For every $q \leq p_0$, there are coboundedly many $\alpha < \omega_1$ for which there exist $q_g, q_h \leq q$ and $g, h \in \mathcal{B}^\omega$ such that*

(1) $\sum_{n=0}^{\infty} g(n) \cdot h(n) = 0$,
(2) $q_g \Vdash f_\alpha(\dot{\dot{x}}) = g$,
(3) $q_h \Vdash f_\alpha(\dot{\dot{x}}) = h$.

Proof: Since $\mathcal{P}$ is σ-closed, we can recursively choose sequences $q_\alpha \in \mathcal{P}$ $(\alpha < \omega_1)$ and $g_\alpha \in \mathcal{B}^\omega$ $(\alpha < \omega_1)$ such that

(48) $q_0 = q$,
(49) $q_\alpha \leq q_\xi$ for all $\xi < \alpha$,
(50) $q_\alpha \Vdash f_\alpha(\dot{\dot{x}}) = g_\alpha$.

Define a $\mathcal{B}$-name $\dot{\Gamma}$ for a subset of ω_1 by

$$\|\omega \cdot \alpha + n \in \dot{\Gamma}\|_{\mathcal{B}} = g_\alpha(n) \quad \text{for all } \alpha, n.$$

Observe that

$$\|\dot\Gamma \text{ is a cofinal branch}\|_{\mathcal{B}} = 1. \tag{51}$$

For if there were $c \in \mathcal{B}$ and $\alpha < \omega_1$ such that $c \Vdash |\dot\Gamma \cap \dot T_\alpha| \neq 1$, then we would have $(q_\alpha, c) \Vdash |\dot{\dot{x}} \cap \dot T_\alpha| \neq 1$. Now by (47) and (51),

$$p_0 \Vdash \|\dot{\dot{x}} \neq \dot\Gamma\|_{\mathcal{B}} = 1. \tag{52}$$

Hence, as $p_0 \Vdash \mathcal{B}$ has the ccc, $p_0 \Vdash \exists \xi < \omega_1 \ \|\forall \alpha \geq \xi \ \dot{\dot{x}} \cap \dot\Gamma \cap \dot T_\alpha = \emptyset\|_{\mathcal{B}} = 1$. And hence there is an $r \leq q$ and a $\xi < \omega_1$ such that

$$r \Vdash \|\dot{\dot{x}} \cap \dot\Gamma \cap \dot T_\alpha = \emptyset\|_{\mathcal{B}} = 1 \quad \text{for all } \alpha \geq \xi. \tag{53}$$

Given $\alpha \geq \xi$, find $s \leq r$ and $h \in \mathcal{B}^\omega$ such that

$$s \Vdash f_\alpha(\dot{\dot{x}}) = h. \tag{54}$$

By (50), (53) and (54), $g_\alpha(n) \cdot h(n) = 0$ for all $n < \omega$, giving the conclusion of the Sublemma. ∎

By applying the Sublemma recursively we can obtain $p_s \in \mathcal{P}$ $(s \in 2^{<\mathbb{N}})$, $g_s \in \mathcal{B}^\omega$ $(s \in 2^{<\mathbb{N}})$ and $\alpha_n < \omega_1$ $(n < \omega)$ such that

(55) $\sum_{n=0}^\infty g_{s^\frown 0}(n) \cdot g_{s^\frown 1}(n) = 0$,

(56) $p_s \Vdash f_{\alpha_{|s|}}(\dot{\dot{x}}) = g_s$,

(57) $p_{s^\frown 0}, p_{s^\frown 1} \leq p_s$.

Put $\alpha = \sup_{n<\omega} \alpha_n$, and for each $\sigma \in 2^\omega$ choose $p_\sigma \in \mathcal{P}$ and $g_\sigma \in \mathcal{B}^\omega$ such that

(58) $p_\sigma \leq p_{\sigma \restriction n}$ for all n,

(59) $p_\sigma \Vdash f_\alpha(\dot{\dot{x}}) = g_\sigma$.

It is easily verified that

$$\sigma \neq \tau \quad \text{implies} \quad \sum_{n=0}^\infty g_\sigma(n) \cdot g_\tau(n) = 0. \tag{60}$$

Since $\sum_{n=0}^\infty g_\sigma(n) = 1$ for all σ, there exists an $\bar n$ such that

$$X = \{\sigma \in 2^\omega : g_\sigma(\bar n) \neq 0\}$$

is uncountable. But then (60) implies that $\{g_\sigma(\bar n) : \sigma \in X\}$ is an uncountable antichain of $\mathcal{B}$, contradicting the fact that $\mathcal{B}$ has the ccc. ∎

Remark 5.10: Suppose now that $\dot{\mathcal{T}}$ is a $\mathcal{B}$-name for a tree of size and height ω_1. While the enumeration in (45) is no longer possible, note that the proof of Sublemma 5.9 did not rely on the fact that $\dot{\mathcal{T}}$ had countable levels. Thus (as is done in [Bau]) we can represent the intersection of a branch with the α^{th} level of $\dot{\mathcal{T}}$ by a member of $\mathcal{B}^{\omega_1}$ with countable support, and use the Sublemma to obtain a complete binary tree of height $\omega + 1$ in $\mathcal{P}$ as above. Then, assuming $\mathfrak{c} > \aleph_1$, we obtain a contradiction with the fact that $\dot{\mathcal{T}}$ names a tree of size $\aleph_1$. This gives a proof of the generalization of Lemma 5.7 with an arbitrary ccc poset in place of a measure algebra. insert here:

Notation: for a cardinal κ and an ordinal θ, we let $\mathcal{C}(\kappa, \theta)$ denote the poset for collapsing θ to an ordinal of cardinality κ by partial functions of cardinality $< \kappa$

Proof of Theorem 5.3: Assume PFA and let $\mathcal{T}$ be an $\mathcal{R}$-name for a tree of size and height ω_1 where $\mathcal{R}$ is some measure algebra. Since PFA implies that $\mathfrak{c} = \aleph_2$ and $2^{\aleph_1} = \aleph_2$ and by Lemma 5.7,

$$\mathcal{C}(\aleph_1, \aleph_2) \Vdash \|\mathcal{T} \text{ has at most } \aleph_1 \text{ cofinal branches}\| = 1 \tag{61}$$

(see also the following discussion). Let $\mathcal{T}^*$ be the $\mathcal{C}(\aleph_1, \aleph_2) \star \mathcal{R}$-name for the tree obtained from applying Lemma 5.4 in the extension by $\mathcal{C}(\aleph_1, \aleph_2)$. In [Lav87] it shown that for every $\mathcal{R}$-name $\mathcal{S}$ for a tree with no branches of length ω_1, there is a ccc poset $\mathcal{L}(\mathcal{S})$ which specializes it. Thus an application of PFA to a suitable family of $\aleph_1$ dense subsets of $\mathcal{C}(\aleph_1, \aleph_2) \star \dot{\mathcal{L}}(\mathcal{T}^*)$ gives an $\mathcal{R}$-name for an essentially specializing map on $\mathcal{T}$. ∎

Let us see why Conjecture 2 now follows provided we can iterate our forcings without adding reals at limit stages. Suppose that $\mathcal{T}$ is an $\mathcal{R}$-name for an ω_1-tree. Since an ω_1-tree is of cardinality $\aleph_1$, any $\mathcal{R}$-name for an ω_1-tree is completely determined by a subalgebra of $\mathcal{R}$ of weight $\aleph_1$. In other words, we may assume that $\mathcal{R} = \mathcal{R}_{\omega_1}$ (the measure algebra of 2^{ω_1}). As we are working under CH, it follows that there are at most $2^{\aleph_1}$ many $\mathcal{R}$-names for branches of $\mathcal{T}$. Thus $\mathcal{C}(\aleph_1, 2^{\aleph_1})$ forces that there are at most $\aleph_1$ many old $\check{\mathcal{R}}$-names for branches of $\mathcal{T}$, and hence by Lemma 5.8 it in fact forces that there are at most $\aleph_1$ many $\check{\mathcal{R}}$-names for these branches. And since $\mathcal{R}$ is absolute for σ-closed forcings, e.g. $\mathcal{C}(\aleph_1, 2^{\aleph_1})$ forces $\mathcal{R} = \check{\mathcal{R}}$ is a measure algebra, after collapsing the hypothesis of Lemma 5.5 is satisfied. Thus the poset $\mathcal{C}(\aleph_1, 2^{\aleph_1}) \star \dot{\mathcal{Q}}(\mathcal{T})$ forces that $\mathcal{T}$ is essentially special with probability one.

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